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A Polyhedral Bound on the Indeterminate Contact Forces in 2D Fixturing and Grasping Arrangements

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Abstract *This paper considers 2D contact arrangements where several bodies grasp, fixture, or support an object via frictional point contacts. Within a strictly rigid body modelling paradigm, when an external wrench (i.e. force and torque) acts on the object, the reaction forces at the contacts are indeterminate and span an unbounded linear space. This paper analyzes the contact forces within a* quasi-rigid body *framework that keeps the desirable geometric properties of rigid body modelling, while also includes more realistic physical effects. Using two principles governing the mechanics of quasi-rigid contacts, we show that for any given external wrench acting on the object, the contact forces lie in a* bounded polyhedral set. *The polyhedral bound depends on the external wrench, the grasp's geometry, and the preload forces. But it does not depend on any detailed knowledge of the contact mechanics parameters. The bound is useful for "robust" grasp and fixture synthesis. Given a collection of external wrenches that may act on an object, the grasp's geometry and preload forces can be chosen such that all of these external wrenches would be automatically supported by the contacts.*

1 Introduction

This paper considers 2D contact arrangements where an object is grasped, fixtured, or supported in static equilibrium by several bodies via frictional point contacts. Under an ideal rigid body assumption, the reaction forces at the contacts due to an external wrench acting on the object are indeterminate and span an unbounded linear space. Generically, in a k-contact planar grasp the indeterminate contact forces span a $(2k-3)$-dimensional linear space. This indeterminacy is an artifact of the simplicity of rigid body models, and it causes difficulties in the analysis, synthesis, and implementation of reliable grasping and fixturing systems. However, in reality all bodies possess some degree of natural compliance due to local material deformation at the contacts. These elastic deformations induce specific contact forces in response to an applied external wrench. Unfortunately, the laws governing compliant deformation depend on various geometric and material properties of the contacting bodies [14]. A detailed knowledge of these properties is not readily available in practice, and measurement of these properties requires sophisticated and time-consuming sensing capabilities.

This paper takes a novel middle-ground approach between rigid body idealization and compliant contact models. We show that when two generic rules governing the mechanics of contact are taken into account, the unbounded linear space of indeterminate forces reduces to a bounded polyhedral set. Moreover, the polyhedral set does not depend on the specific value of the parameters appearing in the contact mechanics rules, thus allowing grasp synthesis under huge uncertainty in the value of these parameters.

Relationship to Prior Work. The indeterminate forces arising from rigid body analysis are in part an artifact of the rigid body modelling assumptions. To measure the actual (unique) reaction forces that arise in an application, thereby resolving the indeterminate forces predicted by off-line analysis, one can install force sensors at the contacts (e.g. [5]). While this approach is useful for controlling the contact forces, it does not provide any analytical insight as to what actual forces might appear in a given application. The analytical study of the indeterminate contact forces has been motivated by power grasp and whole-arm manipulation applications (e.g. [1, 7]). In the context of these applications, the following two approaches were proposed. The first approach implements compliant behavior by some suitable stiffness-control method (e.g. [2, 11]). The stiffness matrix of the resulting closed-loop grasping mechanism can then predict the reaction forces due to external loads acting on the object. A second approach is to measure the joint-torques of the grasping mechanism as a means for resolving the indeterminate forces [1, 9, 10]. Unfortunately, Bicchi [1] has shown that the kinematics of the grasping mechanism induces a subspace of indeterminate forces, which he called the *passive internal forces*, that cannot be measured by joint torques. Omata [8, 9] subsequently reported that under a reasonable assumption of one contact per link, the passive internal forces consist only of tangential forces that lie in a bounded polyhedral set.

The polyhedral bound reported here differs from the one obtained by Omata for power grasps in three fundamental ways. First, we ignore the structure imposed by the kinematics of the grasping mechanism. Rather, we focus on the interaction between the grasped object and its surrounding bodies. The polyhedral bound reported here is consequently useful in applications where joint-torques are not necessarily available, such as fixturing and industrial gripping applications. Second, the third author previously assumed that the contacting bodies are perfectly rigid. In contrast, we assume that the bodies are quasi-rigid and can locally deform at the contacts. Last, the third author previously focused on bounding the tangent component of the indeterminate contact forces. In contrast, we consider both the tangent and normal component of the contact forces.

An important application for the polyhedral bound is the following grasp synthesis approach. We are given a bounded collection of external wrenches that can act on the object, as well as a lower bound on the coefficient of friction at the contacts. Using these two specifications, we select the grasp's geometry and preload forces such that the entire bounded set of indeterminate reaction forces induced by the external wrenches would satisfy the friction-cone constraints at the contacts, even in the presence of great contact parameter uncertainty. The resulting grasp will passively cancel all external wrenches in the given collection, without any of the contacts slipping or breaking away from the object. Note that existing grasp synthesis methods establish preloaded grasps that resist some *unspecified* local neighborhood of external wrenches about the origin (e.g. [15]). In contrast, our synthesis approach generates grasps that resist an entire specified collection of external wrenches.

The paper is structured as follows. In Section 2 we describe two generic principles that govern the mechanics of quasi-rigid contacts: a micro-penetration principle determines the change in the normal component of the contact forces, and a micro-slip principle determines the change in the tangent or frictional component of the contact forces. In Section 3 we derive the polyhedral bound on the contact forces by coupling the effect of the two principles at the individual contacts through the rigid body motions of the grasped object. In Section 4 we demonstrate the polyhedral bound on concrete examples. In Section 5 we describe a grasp synthesis methodology which is based on the polyhedral bound. Finally, we discuss in the concluding section topics for further research.

2 Micro Penetration and Slip

2.1 Grasping Terminology

We assume that a planar object $\mathcal{B}$ is in *frictional point contact* with stationary planar bodies $\mathcal{A}_1, ..., \mathcal{A}_k$ which represent fingertips or fixturing elements. The configuration space (*c-space*) of $\mathcal{B}$ is parametrized by $q = (d_x, d_y, \theta) \in \mathbb{R}^3$ (Figure 1(a)). The velocity of $\mathcal{B}$ at q is represented by a tangent vector $\dot{q}$ which is based at q. Next we review relevant rigid body formulas. Let x_i denote the contact point between $\mathcal{A}_i$ and $\mathcal{B}$, expressed in a fixed world frame. Let r_i denote the same point expressed in $\mathcal{B}$'s body frame (Figure 1(a)). Then x_i is related to r_i by the rigid body transformation: $x_i = X(r_i, q) = R(\theta)r_i + d$, where $R(\theta)$ is the orientation matrix of $\mathcal{B}$. Let $X_{r_i}(q)$ denote the rigid body transformation with r_i held fixed. When $\mathcal{B}$ moves along a c-space curve $q(t)$, the velocity of X_{r_i} is given by $\frac{d}{dt}X_{r_i}(q(t)) = G_i^T \dot{q}(t)$, where G_i^T is the 2×3 Jacobian matrix of X_{r_i}. The Jacobian is given by $G_i^T = [I \;\; J\boldsymbol{\rho}_i]$, where I is a 2×2 identity matrix, $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, and $\boldsymbol{\rho}_i = R(\theta)r_i$.

Contact-Force Space: Let $\boldsymbol{f} = (\boldsymbol{f}_1, ..., \boldsymbol{f}_k)$ denote the contact forces. Then *contact-force space* is the space of tangent and normal components of the contact forces, $((f_1^t, f_1^n), ..., (f_k^t, f_k^n)) \in \mathbb{R}^{2k}$, which is obtained as follows. Let n_i denote the unit normal to the boundary of $\mathcal{A}_i$ and $\mathcal{B}$ at x_i, pointing into $\mathcal{B}$ (Figure 1(a)). Let t_i denote the unit tangent to the boundary of $\mathcal{A}_i$ and $\mathcal{B}$ at x_i. Then $f_i^t = t_i \cdot \boldsymbol{f}_i$ and $f_i^n = n_i \cdot \boldsymbol{f}_i$. The following *tangent and normal projection* matrices project the contact forces onto their tangent and normal coordinates:

$$T = \begin{bmatrix} t_1^T & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & t_k^T \end{bmatrix}_{k \times 2k} \quad N = \begin{bmatrix} n_1^T & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & n_k^T \end{bmatrix}_{k \times 2k}.$$

(Thus $f^t = T\boldsymbol{f}$ and $f^n = N\boldsymbol{f}$, where $f^t = (f_1^t, ..., f_k^t)$ and $f^n = (f_1^n, ..., f_k^n)$.) Finally, the wrench induced on $\mathcal{B}$ by a force $\boldsymbol{f}_i$ acting at x_i, denoted $\boldsymbol{w}_i$, is given by

$$\boldsymbol{w}_i = \begin{pmatrix} \boldsymbol{f}_i \\ (J\boldsymbol{\rho}_i)^T \boldsymbol{f}_i \end{pmatrix} = G_i \boldsymbol{f}_i \quad \text{where } G_i = \begin{bmatrix} I \\ (J\boldsymbol{\rho}_i)^T \end{bmatrix}.$$

When k bodies apply forces $\boldsymbol{f} = (\boldsymbol{f}_1, ..., \boldsymbol{f}_k)$ on $\mathcal{B}$, the net wrench acting on $\mathcal{B}$ is given by $\boldsymbol{w} = \sum_{i=1}^k G_i \boldsymbol{f}_i = G\boldsymbol{f}$, where $G = [G_1 \cdots G_k]_{3 \times 2k}$ is the *grasp matrix.*

2.2 Mechanics of Micro-Penetration

We now formulate a rule for the change in the normal component of a contact force due to elastic deformation at the contact. The usual assumption made in the solid mechanics literature is that the contacting bodies are *quasi-rigid,* meaning that their deformation due to compliance effects is localized to the vicinity of the contacts [14]. This assumption allows us to describe the overall motion of $\mathcal{B}$ relative to the stationary bodies $\mathcal{A}_1, ..., \mathcal{A}_k$ using rigid body kinematics. The quasi-rigidity assumption is valid for all bodies which are not made of exceptionally soft material and do not contain slender substructures.

A convenient lumped-parameter model for the mechanics of compliant contact is based on *overlap*

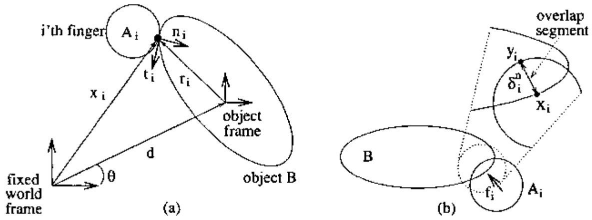

Figure 1: (a) $\mathcal{B}$'s c-space (d_x, d_y, θ). (b) The overlap segment between $\mathcal{A}_i$ and $\mathcal{B}$.

functions [12]. Consider a single contact between $\mathcal{B}$ and $\mathcal{A}_i$. In the absence of deformation, the two bodies contact at a single point. When pushed together, the deformed bodies can be conceptually thought of as overlapping with their undeformed shapes (Fig. 1(b)). Let $\mathcal{B}$ be at a configuration q. Then the *overlap* between $\mathcal{B}$ and $\mathcal{A}_i$, denoted $\delta_i^n(q)$, is *the minimum amount of translation of $\mathcal{B}$ that would separate it from $\mathcal{A}_i$*. For sufficiently small and positive overlap, the *overlap segment* is the unique segment with endpoints on the boundary of $\mathcal{B}$ and $\mathcal{A}_i$ such that the segment is orthogonal to both boundaries. The net normal force induced by the local deformation acts at $\mathcal{B}$'s endpoint of the overlap segment, in the direction of the segment. The magnitude of the net normal force, f_i^n, obeys the generic law:

$$f_i^n = g_i(\delta_i^n) \quad \text{where } g_i'(\delta_i^n) > 0 \text{ when } \delta_i^n > 0. \quad (1)$$

The function g_i is differentiable, $g_i(0) = 0$, and monotonically increasing when $\delta_i^n > 0$. The classical Hertz model [14] establishes that $g_i(\delta_i^n) = \kappa_i(\delta_i^n)^{3/2}$, where κ_i is a specific function of the bodies' local material and geometric properties. However, (1) is valid under more general circumstances than those assumed by the Hertz model. With this background, the principle of micro-penetration is as follows.

Lemma 2.1 (micro-penetration). *Let $\mathcal{B}$ contact a stationary body $\mathcal{A}_i$ s.t. both bodies are quasi-rigid. The change in the normal component of the i^{th} contact force due to an instantaneous motion $\dot{q}$ of $\mathcal{B}$ is:*

$$\Delta f_i^n = -\sigma_i^n (n_i \cdot \dot{X}_{r_i}) = -\sigma_i^n (n_i \cdot G_i^T \dot{q}) \quad \sigma_i^n > 0,$$

where n_i is the inward unit normal to $\mathcal{B}$ at x_i.

The lemma is obtained by application of the chain rule to (1), together with the fact that $\frac{d}{dt}\delta_i^n(q(t)) = -n_i \cdot \dot{X}_{r_i}$ [13]. Intuitively, $-n_i$ points outward with respect to $\mathcal{B}$. If $(-n_i) \cdot \dot{X}_{r_i} > 0$, the overlap between $\mathcal{B}$ and $\mathcal{A}_i$ increases to first-order, and consequently $\Delta f_i^n > 0$. The micro-penetration principle can be written in matrix form as

$$\begin{pmatrix} \Delta f_1^n \\ \vdots \\ \Delta f_k^n \end{pmatrix} = - \begin{bmatrix} \sigma_1^n & & \\ & \ddots & \\ & & \sigma_k^n \end{bmatrix} N G^T \dot{q}, \quad (2)$$

where N is the normal projection matrix, and G is the grasp matrix. Note that (2) *couples* the changes in the normal component of the contact forces through the instantaneous motion $\dot{q}$ of $\mathcal{B}$. Note, too, that this coupling is based on the limited information that the coefficients $\sigma_1^n, \ldots, \sigma_k^n$ have some positive value.

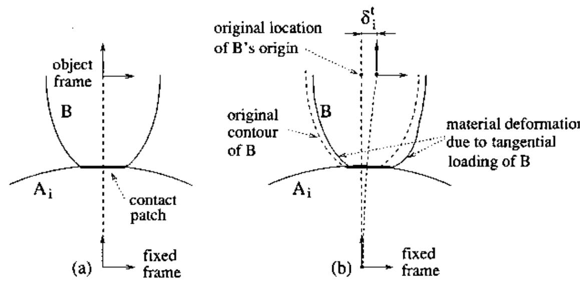

Figure 2: (a) An initial contact patch generated by normal loading. (b) Tangential loading causes tangential displacement of $\mathcal{B}$ without macro-slip.

2.3 Mechanics of Micro-Slip

The principle of micro-slip is based on the tangential compliance induced at a frictional contact by local material deformation [3, 6]. The process underlying this phenomenon is as follows. When two quasi-rigid bodies are preloaded along the normal direction, they locally deform and establish a normal force-field which is distributed along a contact patch (Fig. 2(a)). When the two bodies are next loaded tangentially, they locally deform in a way that generates a tangential force-field which is again continuously distributed along the contact patch (Fig. 2(b)). The usual assumption made in the solid mechanics literature is that *the normal and tangent force-fields interact at the individual points of the contact patch according to Coulomb's law.* Under this assumption, elasticity theory as well as experiments indicate that the tangent force-field consists of two regimes. In an outer ring of the contact area the tangent forces exceed the friction-cone constraint, causing micro-slip at these points. In the complementary inner disc of the contact patch the tangent forces lie within the friction cone, and at these points no micro-slip takes place. As the magnitude of the tangential loading increases the area of the stationary inner-disc shrinks. When the net tangential loading reaches μ times the net normal loading (μ being the coefficient of friction), the inner disc shrinks to a point and the two bodies experience macro-slip at the contact.

We now formulate a rule for the change in the tangent component of the contact force, assuming that the contacting bodies deform but do not slip. In our case $\mathcal{A}_i$ is stationary while $\mathcal{B}$ moves along a c-space curve $q(t)$. Let $\delta_i^t(q(t))$ denote the tangential displacement of $\mathcal{B}$ relative to the i^{th} contact due to motion of $\mathcal{B}$ (Fig. 2(b)). Then the derivative of

δ_i^t along $q(t)$ is the projection of the velocity of the overlap-segment endpoint x_i along the tangent t_i:

$$\frac{d}{dt}\delta_i^t(q(t)) = t_i \cdot \dot{X}_{r_i} = t_i \cdot G_i^T \dot{q}(t). \qquad (3)$$

The net tangent force *opposes* the direction of tangential displacement. Its magnitude, f_i^t, has a dominantly elastic nature which obeys the generic law:

$$f_i^t = h_i(\delta_i^t, \delta_i^n) \quad \text{as long as } f_i^n > 0 \text{ and } |f_i^t| \leq \mu f_i^n.$$

The function h_i is differentiable, $h_i(0, \delta_i^n) = 0$, and for any fixed positive δ_i^n it is monotonically increasing in δ_i^t. Note that f_i^t depends both on δ_i^t and δ_i^n. However, in practice the variation of h_i with respect to δ_i^n is significantly lower than the variation with respect to δ_i^t. Hence we make a simplifying assumption that h_i is approximately $h_i(\delta_i^t, \delta_i^n(q_0))$, where $\delta_i^n(q_0)$ is the normal penetration at the preload configuration q_0. The following lemma summarizes the rule for the change in the tangent component of the contact force.

Lemma 2.2 (micro-slip). *Let $\mathcal{B}$ contact a stationary body $\mathcal{A}_i$, such that both bodies are quasi-rigid. Let the tangent and normal components of the i^{th} contact force satisfy $|f_i^t| \leq \mu f_i^n$ where $f_i^n > 0$. Then the change in the tangent component of the i^{th} contact force due to an instantaneous motion $\dot{q}$ of $\mathcal{B}$ is:*

$$\Delta f_i^t = -\sigma_i^t (t_i \cdot \dot{X}_{r_i}) = -\sigma_i^t (t_i \cdot G_i^T \dot{q}) \quad \sigma_i^t > 0,$$

where t_i is the unit tangent to $\mathcal{B}$ at x_i.

The lemma states that Δf_i^t is proportional to the tangential displacement of $\mathcal{B}$, with the sign of Δf_i^t opposing the direction of tangential displacement. Finally, we write the micro-slip principle in matrix form as

$$\begin{pmatrix} \Delta f_1^t \\ \vdots \\ \Delta f_k^t \end{pmatrix} = -\begin{bmatrix} \sigma_1^t & & \\ & \ddots & \\ & & \sigma_k^t \end{bmatrix} T G^T \dot{q}, \qquad (4)$$

where T is the tangent projection matrix, and G is the grasp matrix. The micro-slip principle (4) *couples* the changes in the tangent contact forces through the motion $\dot{q}$ of $\mathcal{B}$. Moreover, (4) is based on the limited information that $\sigma_1^t, \ldots, \sigma_k^t > 0$.

3 The Polyhedral Bound

The polyhedral bound is derived in contact-force space using the following notation. The coordinates of contact-force space, $(f^t, f^n) \in \mathbb{R}^{2k}$, are the tangent and normal components of the contact forces. We assume an initial preloaded equilibrium grasp of $\mathcal{B}$ at a configuration q_0, and the preload forces components are denoted $(f^t(q_0), f^n(q_0)) \in \mathbb{R}^{2k}$. The changes in the contact forces induced by an external wrench $\boldsymbol{w}_{ext}$ are denoted $(\Delta f^t, \Delta f^n)$, where $\Delta f^t = (\Delta f_1^t, ..., \Delta f_k^t)$ and $\Delta f^n = (\Delta f_1^n, ..., \Delta f_k^n)$.

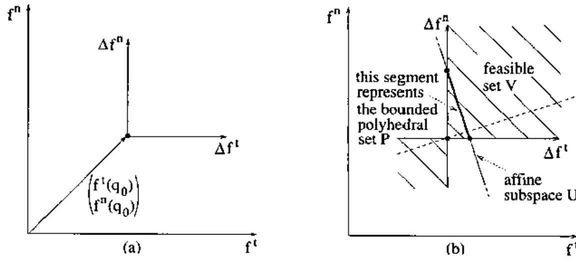

Figure 3: (a) The origin of $(\Delta f^t, \Delta f^n)$-space is determined by the preload forces. (b) The feasible set V fills entire quadrants in $(\Delta f^t, \Delta f^n)$-space.

These changes lie in a linear space called $(\Delta f^t, \Delta f^n)$-*space*, which is a copy of $\mathbb{R}^{2k}$ based at the preload point (Figure 3(a)).

We begin by writing the grasp matrix G as a mapping $\bar{G}$ from contact-force space to the object's wrench space. The contact forces satisfy $\boldsymbol{f} = T^T f^t + N^T f^n$, where T and N are the tangent and normal projection matrices. Pre-multiplying both sides by G gives the expression for $\bar{G}$:

$$G\boldsymbol{f} = \bar{G}\begin{pmatrix} f^t \\ f^n \end{pmatrix}, \quad \text{where } \bar{G} \triangleq G\begin{bmatrix} T \\ N \end{bmatrix}^T \text{ is } 3\times 2k.$$

From now on we refer to $\bar{G}$ as the *grasp matrix*.

Next we write two key expressions using $\bar{G}$. The first expression is the condition for equilibrium induced by the action of $\boldsymbol{w}_{ext}$ on $\mathcal{B}$. The object and surrounding bodies respond to $\boldsymbol{w}_{ext}$ by locally deforming at the contacts. The new forces, $(f^t, f^n) = (f^t(q_0), f^n(q_0)) + (\Delta f^t, \Delta f^n)$, form an equilibrium with $\boldsymbol{w}_{ext}$ according to the linear inhomogeneous equation[1]:

$$\bar{G}\begin{pmatrix} f^t \\ f^n \end{pmatrix} = \bar{G}\begin{pmatrix} \Delta f^t \\ \Delta f^n \end{pmatrix} = \boldsymbol{w}_{ext}, \qquad (5)$$

where we used the fact that the $\bar{G}\left(\begin{smallmatrix} f^t(q_0) \\ f^n(q_0) \end{smallmatrix}\right) = \vec{0}$. The solutions $(\Delta f^t, \Delta f^n)$ to (5) form an *affine subspace* denoted U (Fig. 3(b)). Next we parametrize U in terms of the null space of $\bar{G}$. Let m be the dimension of the kernel of $\bar{G}$, where generically $m = 2k - 3$. Let E be the $2k{\times}m$ matrix whose columns span the kernel of $\bar{G}$. We also need the particular solution of (5) given by $(\Delta f^t, \Delta f^n) = \bar{G}^\dagger \boldsymbol{w}_{ext}$, where $\bar{G}^\dagger = \bar{G}^T[\bar{G}\bar{G}^T]^{-1}$ is the 3×3 pseudo-inverse of $\bar{G}$. Using this particular solution, U is parametrized by

$$U = \left\{ \begin{pmatrix} \Delta f^t \\ \Delta f^n \end{pmatrix} = E\boldsymbol{\nu} + \bar{G}^\dagger \boldsymbol{w}_{ext} : \boldsymbol{\nu} \in \mathbb{R}^m \right\}. \quad (6)$$

The second key expression specifies what changes in the contact forces are generated by instantaneous motions of $\mathcal{B}$. We call this collection of contact-force changes the *feasible set* V. Using the micro-

[1]Eq. (5) holds for the *negated* wrench $-\boldsymbol{w}_{ext}$. Subsequent mentions of the external wrench therefore refer to $-\boldsymbol{w}_{ext}$.

penetration and micro-slip principles, the contact-force change induced by a motion $\dot{q}$ of $\mathcal{B}$ is:

$$\begin{pmatrix} \Delta f^t \\ \Delta f^n \end{pmatrix} = -\begin{bmatrix} \Sigma_t & 0 \\ 0 & \Sigma_n \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix} G^T \dot{q},$$

where $\Sigma_t = \mathsf{diag}(\sigma_1^t, ..., \sigma_k^t)$ and $\Sigma_n = \mathsf{diag}(\sigma_1^n, ..., \sigma_k^n)$. Since $\bar{G} = G\begin{bmatrix} T \\ N \end{bmatrix}^T$, the feasible set is given by

$$V = \left\{ \begin{pmatrix} \Delta f^t \\ \Delta f^n \end{pmatrix} = -\Sigma \bar{G}^T \dot{q} : \dot{q} \in \mathbb{R}^3, \Sigma > 0 \right\}, \quad (7)$$

where $\Sigma = \mathrm{diag}(\Sigma_t, \Sigma_n)$, and $\Sigma > 0$ means that the diagonal entries of Σ are *unspecified positive parameters.* The set of indeterminate contact forces due to $\boldsymbol{w}_{ext}$, denoted $\mathcal{P}$, is given by $\mathcal{P} = U \cap V$.

Our task is to show that $\mathcal{P}$ is a bounded polyhedral set. We first argue that $\mathcal{P}$ is polyhedral. Let a *p-quadrant* in $\mathbb{R}^p$ be the rectangular cone spanned by a particular choice of directions along the coordinate-axes of $\mathbb{R}^p$. It is shown in Ref. [13] that the feasible set V is a union of entire $2k$-quadrants in $(\Delta f^t, \Delta f^n)$-space (Figure 3(b)). Hence V is an unbounded polyhedral set. On the other hand, U is an affine subspace in $(\Delta f^t, \Delta f^n)$-space. It follows that the intersection $\mathcal{P} = U \cap V$ is a polyhedral set. However, it still remains to show the key result that $\mathcal{P}$ is bounded.

Proposition 3.1 (Boundedness). *The set of contact-force changes induced by the action of $\boldsymbol{w}_{ext}$ on $\mathcal{B}$, $\mathcal{P} = U \cap V$, is bounded in $(\Delta f^t, \Delta f^n)$-space.*

Proof: First we convert the parametrized representation (7) of V into an implicit representation. Since the kernel of $\bar{G}$ is spanned by the columns of E, multiplying both sides of (7) by $E^T \Sigma^{-1}$ gives:

$$\begin{pmatrix} \Delta f^t \\ \Delta f^n \end{pmatrix} \in V \implies E^T \Sigma^{-1} \begin{pmatrix} \Delta f^t \\ \Delta f^n \end{pmatrix} = \vec{0}. \quad (8)$$

Next we substitute into (8) the null-space parametrization (6) of U,

$$E^T \Sigma^{-1} (E\boldsymbol{\nu} + \bar{G}^\dagger \boldsymbol{w}_{ext}) = \vec{0}. \quad (9)$$

The solutions $\boldsymbol{\nu}$ of (9) parametrize the set $\mathcal{P}$. Any $\boldsymbol{\nu} \in \mathbb{R}^m$ can be written as $\boldsymbol{\nu} = \nu \hat{\boldsymbol{\nu}}$, where ν and $\hat{\boldsymbol{\nu}}$ denote the magnitude and direction of $\boldsymbol{\nu}$. Our objective is to show that the magnitude of the solutions $\boldsymbol{\nu}$ to (9) is bounded. Multiplying both sides of (9) by $\hat{\boldsymbol{\nu}}$ gives

$$(\hat{\boldsymbol{\nu}}^T E^T \Sigma^{-1} E \hat{\boldsymbol{\nu}}) \nu + \hat{\boldsymbol{\nu}}^T E^T \Sigma^{-1} \bar{G}^\dagger \boldsymbol{w}_{ext} = 0. \quad (10)$$

Let $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_{2k}) \in \mathbb{R}^{2k}$ denote the positive parameters along the diagonal of Σ. Then (10) can be solved for ν as a function of $\hat{\boldsymbol{\nu}}$ and $\boldsymbol{\sigma}$,

$$\nu(\hat{\boldsymbol{\nu}}, \boldsymbol{\sigma}) = -\frac{\hat{\boldsymbol{\nu}}^T E^T \Sigma^{-1}(\boldsymbol{\sigma}) \bar{G}^\dagger \boldsymbol{w}_{ext}}{\hat{\boldsymbol{\nu}}^T E^T \Sigma^{-1}(\boldsymbol{\sigma}) E \hat{\boldsymbol{\nu}}}. \quad (11)$$

Note that $\nu(\hat{\boldsymbol{\nu}}, \boldsymbol{\sigma})$ is well defined, since Σ and $E^T \Sigma^{-1} E$ are positive-definite matrices when $\sigma_1, ..., \sigma_{2k} > 0$. Let $\hat{\boldsymbol{\sigma}}$ denote a unit magnitude vector in $\boldsymbol{\sigma}$-space, and let $\alpha(s) = s\hat{\boldsymbol{\sigma}}$ be a line through the origin with direction $\hat{\boldsymbol{\sigma}}$ in $\boldsymbol{\sigma}$-space. Then $\Sigma^{-1}(s\hat{\boldsymbol{\sigma}}) = \frac{1}{s}\Sigma^{-1}(\hat{\boldsymbol{\sigma}})$ and $[E^T \Sigma^{-1}(s\hat{\boldsymbol{\sigma}}) E]^{-1} = s[E^T \Sigma^{-1}(\hat{\boldsymbol{\sigma}}) E]^{-1}$. Substituting these relations in (11), we obtain that $\nu(\hat{\boldsymbol{\nu}}, \boldsymbol{\sigma})$ attains constant values along lines through the origin in $\boldsymbol{\sigma}$-space:

$$\nu(\hat{\boldsymbol{\nu}}, s\hat{\boldsymbol{\sigma}}) = -\frac{\hat{\boldsymbol{\nu}}^T E^T \Sigma^{-1}(\hat{\boldsymbol{\sigma}}) \bar{G}^\dagger \boldsymbol{w}_{ext}}{\hat{\boldsymbol{\nu}}^T E^T \Sigma^{-1}(\hat{\boldsymbol{\sigma}}) E \hat{\boldsymbol{\nu}}}. \quad (12)$$

Thus it suffices to evaluate ν over $(\hat{\boldsymbol{\nu}}, \hat{\boldsymbol{\sigma}}) \in \mathbb{R}^m \times \mathbb{R}^{2k}$ such that $\sigma_1, ..., \sigma_{2k} > 0$. Let $\mathcal{S}$ denote the closure of this set, $\mathcal{S} = \{(\hat{\boldsymbol{\nu}}, \hat{\boldsymbol{\sigma}}) \in \mathbb{R}^m \times \mathbb{R}^{2k} : \sigma_1, ..., \sigma_{2k} \geq 0\}$. It is shown in Ref. [13] that ν is a continuous function on $\mathcal{S}$. Since $\mathcal{S}$ is compact and a continuous function is bounded on a compact set, $\nu(\hat{\boldsymbol{\nu}}, \hat{\boldsymbol{\sigma}})$ is bounded over $\mathcal{S}$. In particular, ν is bounded over the interior of $\mathcal{S}$ where $\sigma_1, .., \sigma_{2k} > 0$. Finally, $\mathcal{P}$ consists of $\begin{pmatrix} \Delta f^t \\ \Delta f^n \end{pmatrix} = E\boldsymbol{\nu} + \bar{G}^\dagger \boldsymbol{w}_{ext}$ s.t. $\boldsymbol{\nu}$ is a solution of (9). Since the solutions $\boldsymbol{\nu}$ of (9) are bounded, $\mathcal{P}$ is bounded. □

The following theorem summarizes the results concerning the indeterminate contact forces.

Theorem 1. *Let k planar bodies hold a 2D object $\mathcal{B}$ in an equilibrium grasp, with preload forces $(f^t(q_0), f^n(q_0))$. Let an external wrench $\boldsymbol{w}_{ext}$ act on $\mathcal{B}$, such that none of the contacts breaks or slips. Then the contact forces induced by $\boldsymbol{w}_{ext}$ are given by*

$$\begin{pmatrix} f^t \\ f^n \end{pmatrix} = \begin{pmatrix} f^t(q_0) \\ f^n(q_0) \end{pmatrix} + \begin{pmatrix} \Delta f^t \\ \Delta f^n \end{pmatrix} \quad \begin{pmatrix} \Delta f^t \\ \Delta f^n \end{pmatrix} \in \mathcal{P},$$

where $\mathcal{P}$ is a **bounded polyhedral set.** *Moreover, the dimension of $\mathcal{P}$ is generically $m = 2k-3$.*

The polyhedral bound reduces the infinite linear space of indeterminate forces to a bounded polyhedral subset. Moreover, the bound does not depend on the specific value of the contact parameters σ_i^t and σ_i^n $(i = 1, \ldots, k)$, and is thus useful for robust grasp synthesis. We conclude with a mention of three properties of the polyhedral bound [13]. First, $\mathcal{P}$ is a union of *convex* polyhedra, each contained in a particular $2k$-quadrant of $(\Delta f^t, \Delta f^n)$-space. Second, the planar faces of these polyhedra are embedded in the coordinate hyperplane of $(\Delta f^t, \Delta f^n)$-space. This means that each planar face of $\mathcal{P}$ is associated with the vanishing of a tangent or normal force-component at one of the contacts. Last, the number of vertices in $\mathcal{P}$ is polynomial in the number of contacts, and can be computed in polynomial time [13].

4 Examples of the Bound

In the following examples we assume that none of the contacts breaks or slips in response to the applied external wrench. As discussed below, the validity of this assumption requires a selection of sufficiently high preload forces.

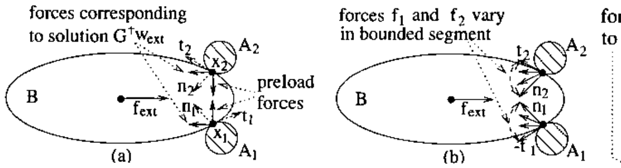

Figure 4: (a) The preload forces and solution $\bar{G}^{\dagger}\boldsymbol{w}_{ext}$. (b) The forces of the bounded segment $\mathcal{P}$.

Example 1. Consider an ellipse held by two disc fingers as shown in Figure 4(a). The ellipse is subjected to a horizontal external force, $\boldsymbol{f}_{ext}$, that acts at the ellipse's center against the fingers. The grasp matrix $\bar{G}$ is 3×4 and $m = 1$. It follows from Theorem 1 that the collection of indeterminate contact forces induced by $\boldsymbol{f}_{ext}$ is a *bounded segment* in contact-force space. The kernel of $\bar{G}$ consists of all forces that act at x_1 and x_2 with equal magnitude and opposite directions. The kernel is given in contact-force space by $\nu(1, 1, -1, 1)$, where ν is an arbitrary scalar. The solution $\bar{G}^{\dagger}\boldsymbol{w}_{ext}$ consists of two horizontal forces given by $\bar{G}^{\dagger}\boldsymbol{w}_{ext} = \frac{f_{ext}}{2\sqrt{2}}(-1, 1, 1, 1)$, where f_{ext} denotes the magnitude of $\boldsymbol{f}_{ext}$. Thus, the set $\mathcal{P}$ corresponds to solutions ν of the equation:

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \nu + \frac{f_{ext}}{2\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = - \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_4 \end{bmatrix} \bar{G}^T \dot{q},$$

where $\dot{q} \in I\!R^3$ and $\sigma_1, ..., \sigma_4 > 0$. As stated above, the endpoints of $\mathcal{P}$ lie on the coordinate hyperplanes in $(\Delta f^t, \Delta f^n)$-space. Equating the four scalar equations on the left-side with zero, we obtain two pairs of identical equations: $\nu - \frac{f_{ext}}{2\sqrt{2}} = 0$ and $\nu + \frac{f_{ext}}{2\sqrt{2}} = 0$. The bounded segment $\mathcal{P}$ is thus given by

$$\mathcal{P} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \nu + \frac{f_{ext}}{2\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} : -\frac{f_{ext}}{2\sqrt{2}} \le \nu \le \frac{f_{ext}}{2\sqrt{2}} \right\}.$$

The forces of $\mathcal{P}$ are depicted in Figure 4(b).

Example 2. Consider a triangular object held by three disc fingers along the contact normals (Fig. 5). The object is subjected to an external torque τ_{ext} that acts about its center. Contact-force space is six-dimensional and $\bar{G}$ is 3×6. The kernel of $\bar{G}$ is three-dimensional and is spanned by the following vectors. Every pair of contacts contributes one basis vector, given by two forces that act in opposite directions along the line connecting the two contacts. The tangent and normal components of the three basis vectors comprise the columns of the 6×3 matrix E written below. Next consider the solution $\bar{G}^{\dagger}\boldsymbol{w}_{ext}$. Let ρ denote the distance from the object's center to the three contacts. Then $\bar{G}^{\dagger}\boldsymbol{w}_{ext}$ is given by three tangent forces of equal magnitude, which together

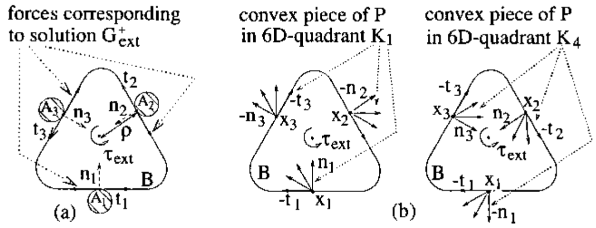

Figure 5: (a) The preload forces and solution $\bar{G}^{\dagger}\boldsymbol{w}_{ext}$. (b) The forces corresponding to the portions of $\mathcal{P}$ in the $6D$-quadrants K_1 and K_4.

balance τ_{ext} (Fig. 5(a)). The expressions for E and $\bar{G}^{\dagger}\boldsymbol{w}_{ext}$ are written in the following equation, that represents the intersection of U with V:

$$\begin{pmatrix} 0 & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} - \frac{\tau_{ext}}{3\rho} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = -\Sigma \bar{G}^T \dot{q},$$

where $\dot{q} \in I\!R^3$ and $\Sigma = \mathsf{diag}(\sigma_1, ..., \sigma_6) > 0$. The feasible set V fills entire $6D$-quadrants in $(\Delta f^t, \Delta f^n)$-space. Let $(\pm 1, ..., \pm 1)$ denote the $2^6 = 64$ possible $6D$-quadrants in this space. Then an analysis omitted here reveals that V occupies 24 of the $6D$-quadrants, while U intersects six of these quadrants. The six $6D$-quadrants are: $K_1 = (-1, 1, -1, -1, -1, -1)$, $K_2 = (-1, -1, -1, 1, -1, -1)$, $K_3 = (-1, -1, -1, -1, -1, 1)$, $K_4 = (-1, -1, -1, 1, -1, 1)$, $K_5 = (-1, 1, -1, -1, -1, 1)$, $K_6 = (-1, 1, -1, 1, -1, -1)$. Each K_i determines one convex piece of $\mathcal{P}$, and the entire polyhedral set is given by $\mathcal{P} = \cup_{i=1}^{6} U \cap K_i$. The forces of $\mathcal{P}$ in K_1 and K_4 are depicted in Figure 5(b). The forces in $K_3, \ldots, K_6$ are cyclic permutations of the ones depicted for K_1 and K_4.

5 Robust Grasp Synthesis

This section briefly describes an application of the polyhedral bound to grasp synthesis. Our synthesis approach accepts as inputs a bounded collection of external wrenches that can act on the object, denoted $\mathcal{W}$, and a lower bound on the coefficient of friction, denoted μ. Let $\mathcal{P}_w$ denote the set of contact-force changes induced by $\boldsymbol{w} \in \mathcal{W}$. The indeterminate forces induced by $\boldsymbol{w}$ form a bounded polyhedral set with base-point at the preload forces, $f(q_0) = (f^t(q_0), f^n(q_0))$. This set is denoted $f(q_0) + \mathcal{P}_w$.

First consider the selection of preload forces such that none of the contacts will break in response to $\boldsymbol{w}$. Let H be the set of contact forces with non-negative normal component: $H = \left\{ \begin{pmatrix} f^t \\ f^n \end{pmatrix} = \begin{pmatrix} f^t(q_0) \\ f^n(q_0) \end{pmatrix} + \begin{pmatrix} \Delta f^t \\ \Delta f^n \end{pmatrix} : f^n(q_0) + \Delta f^n \ge \vec{0} \right\}$. If $f(q_0) + \mathcal{P}_w$ is completely contained in H, none of the contacts would break in response to $\boldsymbol{w}$. For purposes of grasp synthesis, we assert the following fact. *For any given $\boldsymbol{w} \in \mathcal{W}$, there exist sufficiently high preload forces such that $f(q_0)+\mathcal{P}_w$ is completely contained in H.*

Next consider the selection of preload forces such that none of the contacts will slip in response to $\boldsymbol{w}$. Let FC_i be the set of contact forces associated with the i^{th} friction cone: $FC_i=\{(f_i^t,f_i^n): |f_i^t(q_0)+\Delta f_i^t| \le \mu(f_i^n(q_0)+\Delta f_i^n)\}$ where $(f_i^t,f_i^n)=(f_i^t(q_0),f_i^n(q_0))+(\Delta f_i^t,\Delta f_i^n)$, and let $FC = \cap_{i=1}^k FC_i$. Recall now that an equilibrium grasp is *force closure* iff the grasping forces lie in the interior of the friction cones. This condition implies the following result [13]. *If a grasp is force closure, then for any $\boldsymbol{w}\in\mathcal{W}$ there exist sufficiently high preload forces such that $f(q_0)+\mathcal{P}_w$ is completely contained in FC.* The above two results imply that we may freely select the grasp's geometry, then adjust the preload forces such that none of the contacts breaks or slips in response to a collection $\mathcal{W}$ of external wrenches.

The following result provides an efficient means for computing the contact-force changes induced by $\mathcal{W}$. The set $\mathcal{P}_w$ is a union of convex polyhedra, each contained in a particular $2k$-quadrant in $(\Delta f^t,\Delta f^n)$-space. Let K be one such $2k$-quadrant. Then the following result holds true [13]. *If $\mathcal{W}$ is a small convex neighborhood about a nominal external wrench $\boldsymbol{w}_0$, the collection of force-changes in K is given by the convex hull: $\cup_{\boldsymbol{w}\in\mathcal{W}}\mathcal{P}_w \cap K = Conv\{\mathcal{P}_{\boldsymbol{w}_1}\cap K,\ldots,\mathcal{P}_{\boldsymbol{w}_q}\cap K\}$, where $\boldsymbol{w}_1,\ldots,\boldsymbol{w}_q$ are the vertices of $\mathcal{W}$.* To summarize, we first compute the polyhedral sets $\mathcal{P}_{\boldsymbol{w}_1},\ldots,\mathcal{P}_{\boldsymbol{w}_q}$. The convex hull of these sets in the $2k$-quadrants K gives the polyhedral set of force changes due to $\mathcal{W}$. The latter set has a base-point at the preload forces, $f(q_0)$. But the friction cone constraints are linear inequalities in $f(q_0)$. Hence the selection of preload forces such that the bounded set of indeterminate reaction forces induced by $\mathcal{W}$ satisfies the friction-cone constraints is a linear programming problem. The selection of suitable preload forces is illustrated with examples in Ref. [13].

6 Conclusion

The quasi-rigid body framework allows the use of two generic principles that determine the normal and tangent force changes due to local material deformation. Using these principles, we obtained that the force changes must lie in a *bounded polyhedral set.* The polyhedral set does not depend on specific knowledge of the contact parameters, and is thus useful for robust grasp synthesis. Using the preload forces as a design parameter, we described a grasp synthesis approach for selecting the preload forces such that an entire collection of external wrenches would be automatically resisted by the preloaded grasp. Topics currently under investigation include extension of the polyhedral bound to 3D grasps, and inclusion of the contact points location as a second design parameter in our grasp synthesis method. Finally, we are in the process of constructing an experimental fixturing system for testing our theoretical predictions [4].

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